

Ferromagnetism in a hard-core boson model

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Abstract. The problem of ferromagnetism – associated with a ground state with maximal total spin – is discussed in the framework of a hard-core model, which forbids the occupancy at each site with more than one particle. It is shown that the emergence of ferromagnetism on finite square lattices crucially depends on the statistics of the particles. Fermions (electrons) lead to the well-known instabilities for finite hole densities, whereas for bosons (with spin) ferromagnetism appears to be stable for all hole densities.

PACS. 71.10.Fd Lattice fermion models (Hubbard model, etc.) – 71.27.+a Strongly correlated electron systems; heavy fermions – 75.10.Jm Quantized spin models

1 Introduction

It is generally believed that ferromagnetism should be explained in the framework of a Hamiltonian for strongly coupled electrons. Such a Hamiltonian on a lattice has been proposed 40 years ago by Hubbard [1–3]. It contains a nearest neighbour hopping term H_t and an on-site Coulomb repulsion (of strength U) H_U for the electrons.

Both terms are “blind” to the spin of the electrons and it is therefore far from being obvious that such a model should lead to ferromagnetism. The limit $U \rightarrow \infty$ – here called “hard-core condition” – enforces that each site can be occupied by at most one electron. In this limit, the number Q of electrons which can be accommodated on a lattice of N sites is limited $Q \leq N$. In the charge sector $Q = N$ every spin configuration of the electrons is a possible ground state. On a square lattice this degeneracy is lifted in the sector $Q = N - 1$ with one hole. Nagaoka [4] constructed the ground state and found that the total spin $S(Q)$ of the electrons is maximal:

$$S(Q) = \frac{Q}{2}, \quad Q = N - 1 \quad (1.1)$$

a clear signature for ferromagnetism.

There has been an intensive search for ferromagnetism in the $t - J$ model with $N_h = N - Q$ holes. Numerical results on finite clusters [5,6] at vanishing exchange coupling ($\alpha = 0$, see Eq. (2.1)) show that the ground state does not have maximal spin except for the Nagaoka case $Q = N - 1$. The instability of the Nagaoka ferromagnetic

state in sectors with 2 and more holes has been investigated with variational ground states of the Gutzwiller type [7–10]. It was found that there exists a critical hole density $\delta = N_h/N = 1 - \rho$ such that the ground states cannot be ferromagnetic for $\delta > \delta_{crit.}$. Improvements of the variational ansatz led to a successive decrease of $\delta_{crit.}$ for the $2D$ case

$$\delta_{crit.} = 0.49^{[7]}, 0.41^{[8]}, 0.29^{[9,10]}, 0.251^{[11]}.$$

On the other hand Barbieri et al. [12] have argued that the ferromagnetic ground state emerges in the thermodynamical limit $N \rightarrow \infty$, if the hole density vanishes ($\delta \rightarrow 0$). The existence of a lower bound $\delta > \delta_{crit.}$ for the hole density δ , where ferromagnetism is not possible, is indeed a quite general feature, which has been found also in higher dimensions and for various types of lattices [13]. An introduction to ferromagnetism in the Hubbard model can be found in reference [14].

In this situation one might ask for the reason why hard-core models with fermions (i.e. electrons) fail to “explain” ferromagnetism on finite clusters (except for the 1 hole case). We want to demonstrate in this paper that this failure can be traced back to the anticommutation relations for the fermion operators. For this purpose, we substitute in the $t - J$ model the fermionic degrees of freedom by bosonic ones. In order to facilitate the comparison of the fermionic and bosonic version, we assume that the bosons carry here as well spin 1/2 and experience the same anti-ferromagnetic interaction [15].

Therefore the Hamiltonians for a hard-core model with bosons (b) and fermions (f) are both blind with respect to the spin. Nevertheless, we find pronounced differences (and similarities) in the ground

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state energies $E_i(Q, S_i(Q))$, $i = f, b$ and ground state spins $S_i(Q)$, $i = f, b$.

(a) For $Q = 0, 1, N - 1$ the ground state energies and ground state spins coincide

$$E_b(Q, S_b(Q)) = E_f(Q, S_f(Q)) \quad (1.2)$$

$$S_b(Q) = S_f(Q) = Q/2. \quad (1.3)$$

This means in particular that the construction of the Nagaoka ferromagnet state holds for fermions and for bosons. The restriction to the one hole sector $N_h = 1$, $Q = N - 1$ is crucial. Moreover one observes a hole-particle symmetry for $Q = 1$,

$$E_i(Q, S_i(Q)) = E_i(N - Q, S_i(N - Q)) \quad i = f, b. \quad (1.4)$$

(b) For $Q = 2$ the ground state energies coincide [(1.2) for $Q = 2$], but the ground state spins are different

$$S_f(Q = 2) = 0 \quad S_b(Q = 2) = 1, \quad (1.5)$$

i.e. in the bosonic case the ground state spin is maximal.

(c) For $Q = 3, \dots, N - 2$ ground state energies and ground state spins are different for the fermionic and bosonic version. In the bosonic version the ground state spin $S_b(Q) = Q/2$ is maximal as shown in Appendix A. The ground state energies $E_b(Q, S_b(Q))$ show the hole-particle symmetry (1.4) for these Q -values. This symmetry is not present in the fermionic version. In the latter case the ground state spin $S_f(Q)$ is “erratic” [5].

We think that these properties are interesting enough to justify a detailed study of the ground state properties of the bosonic $t - J$ model and to compare its properties with the corresponding properties in the fermionic version.

The outline of the paper is the following:

In Section 2 we review the definition of the $t - J$ model and point out the differences between the fermionic and bosonic version. Consequences for the ground state energies on the smallest (2×2) cluster are discussed in Section 3. The phase diagram of the bosonic $t - J$ model in dimensions $D = 2$ and $D = 1$ is discussed in Sections 4 and 5, respectively.

2 The bosonic version of the $t - J$ model in two dimensions

Let us first recall the definition of the $t - J$ Hamiltonian with fermionic degrees of freedom:

$$H = t(H_t + \alpha H_J) \quad \alpha = J/t. \quad (2.1)$$

The hopping term

$$H_t = -\mathcal{P} \sum_{\langle x, y \rangle} \sum_{\sigma} (c_{\sigma}^{\dagger}(x) c_{\sigma}(y) + \text{h.c.}) \mathcal{P} \quad (2.2)$$

and the spin exchange part

$$H_J = \mathcal{P} \sum_{\langle x, y \rangle} \left(\mathbf{S}(x) \mathbf{S}(y) - \frac{1}{4} n(x) n(y) \right) \mathcal{P} \quad (2.3)$$

can be expressed in terms of creation ($c_{\sigma}^{\dagger}(x)$) and annihilation ($c_{\sigma}(x)$) operators for the electrons, which obey anticommutation relations

$$\begin{aligned} \{c_{\sigma}^{\dagger}(x), c_{\sigma'}^{\dagger}(x')\} &= \{c_{\sigma}(x), c_{\sigma'}(x')\} = 0 \\ \{c_{\sigma}^{\dagger}(x), c_{\sigma'}(x')\} &= \delta_{\sigma, \sigma'} \delta_{x, x'}; \end{aligned} \quad (2.4)$$

$$n_{\sigma}(x) = c_{\sigma}^{\dagger}(x) c_{\sigma}(x) \quad (2.5)$$

is the number operator for an electron at site x with spin σ . Owing to the “hard-core condition” which is imposed by the projector \mathcal{P} double occupancy is forbidden i.e. $\sum_{\sigma} n_{\sigma}(x) = 0, 1$. The latter can be derived from a Hubbard model [1] with infinite on-site Coulomb repulsion ($U/t \rightarrow \infty$).

The construction of a state with Q electrons

$$|x_1^{\sigma_1}, x_2^{\sigma_2}, \dots, x_Q^{\sigma_Q}\rangle = c_{\sigma_1}^{\dagger}(x_1) \dots c_{\sigma_Q}^{\dagger}(x_Q) |0\rangle \quad (2.6)$$

by application of creation operators to the vacuum $|0\rangle$ demands an ordering of all sites on the 2 dimensional square lattice. Owing to the anticommutation rules (2.4) a different ordering (e.g. with $x_1^{\sigma_1}, x_2^{\sigma_2}$ interchanged) leads to a state which might differ from the former one in sign. In the following we will denote the traditional model (2.1)–(2.3) with fermions as “fermionic $t - J$ model”.

Let us now turn to the “bosonic $t - J$ model”, which we simply define by substituting the anticommuting creation and annihilation operators

$$c_{\sigma}^{\dagger}(x) \rightarrow a_{\sigma}^{\dagger}(x) \quad c_{\sigma}(x) \rightarrow a_{\sigma}(x) \quad (2.7)$$

by commuting ones:

$$\begin{aligned} [a_{\sigma}^{\dagger}(x), a_{\sigma'}^{\dagger}(x')] &= [a_{\sigma}(x), a_{\sigma'}(x')] = 0 \\ [a_{\sigma}^{\dagger}(x), a_{\sigma'}(x')] &= \delta_{\sigma, \sigma'} \delta_{x, x'}. \end{aligned} \quad (2.8)$$

The eigenvalues of the number operator

$$n_{\sigma}(x) = a_{\sigma}^{\dagger}(x) a_{\sigma}(x) \quad (2.9)$$

are restricted again to $n_{\sigma}(x) = 0, 1$ due to the “hard-core condition”. This model has been discussed first in references ([16–19]).

Quite recently a generalized Hubbard model with fermionic and bosonic degrees of freedom has been proposed and investigated in order to study a mixture of ultracold bosonic and fermionic atoms in an optical lattice [20].

The construction of a state with Q bosons by application of creation operators on the vacuum, however, is symmetric under the permutation of sites – due to the commutation relations (2.8). There does not exist a “sign problem”. This is the only difference between the fermionic and bosonic version of the $t - J$ model [cf. Ref. (16)]. One can easily verify, that the action of the spin exchange term (2.3) is the same in the fermionic and bosonic version. On the other hand the hopping term H_t acts indeed in a different way on the fermionic and bosonic states. In the latter case H_t can be expressed in terms of

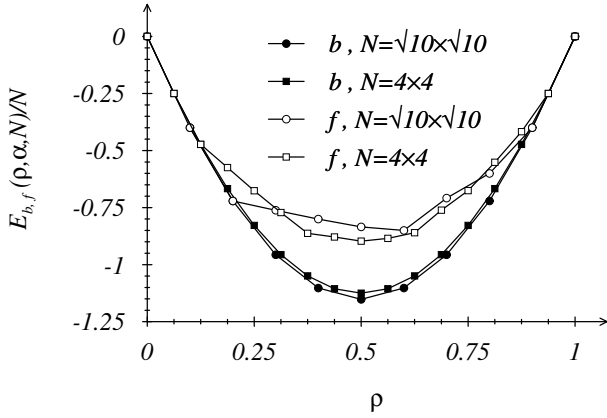


Fig. 1. Ground state energies per site of a periodic 4×4 and a $\sqrt{10} \times \sqrt{10}$ lattice for both fermionic and bosonic $t - J$ models at $\alpha = 0$.

nearest neighbour permutations $P(x, y)$ which interchange a particle and a hole at sites x and y :

$$H_t = - \sum_{\langle x, y \rangle} P(x, y) \left(n(x)n_h(y) + n(y)n_h(x) \right). \quad (2.10)$$

Here $n(x) = n_+(x) + n_-(x) = 0, 1$ is the number of bosons at site x .

In Appendix A the Hamiltonian (2.10) is proven to have a ferromagnetic ground state for all Q . Moreover it is shown that the ground state energies of the bosonic $t - J$ model at $\alpha = 0$ are symmetric under the particle-hole transformation

$$E_b(Q, \alpha = 0, N) = E_b(N - Q, \alpha = 0, N) \quad (2.11)$$

as can be seen in Figure 1.

Here we also show the ground state energy for hard-core fermions, where the hole-particle symmetry (2.11) is lost (except for the Nagaoka case $Q = N - 1$)

$$E_f(Q, \alpha = 0, N) \neq E_f(N - Q, \alpha = 0, N). \quad (2.12)$$

Moreover the ground state energies of the hard-core bosons are below those of the hard-core fermions

$$E_b(Q, \alpha = 0, N) \leq E_f(Q, \alpha = 0, N). \quad (2.13)$$

Comparing the results from lattices $L \times L$, $L = \sqrt{10}, 4$ we find that the finite-size dependence is rather small and smooth in the bosonic case, in contrast to the fermionic case. The result presented in Figure 1 were obtained with a Lanczos algorithm. Concerning the fermionic version, they agree with reference [5] (Riera, Young [22]).

In both versions (fermionic and bosonic) the $t - J$ Hamiltonian (2.1) conserves the total charge

$$Q = \sum_x (n_+(x) + n_-(x)) \quad (2.14)$$

and total spin

$$\mathbf{S} = \frac{1}{2} \sum_x \sigma(\mathbf{x}) \quad (2.15)$$

such that the eigenvalues of $\mathbf{S}^2 = S(S + 1)$ can be used to characterize the eigenstates of the Hamiltonian. Ferromagnetic eigenstates $|F, Q\rangle$ have maximal spin $S = Q/2$. These states are simultaneous eigenstates of the hopping part H_t and the spin coupling part H_J

$$tH_t|F, Q\rangle = E_F(Q)|F, Q\rangle \quad (2.16)$$

$$H_J|F, Q\rangle = 0 \quad (2.17)$$

such that the eigenvalues $E_F(Q)$ for the $t - J$ Hamiltonian (2.1) are independent of α . This is a consequence of the fact that the hopping part H_t is blind with respect to the spin of the electrons. There is no difference in the hopping of spin-up and spin-down particles. On the other hand, the ferromagnetic eigenstates $|F, Q\rangle$ are totally symmetric under any permutation of spin-up and spin-down particles. Therefore, the application of the nearest neighbour spin exchange coupling

$$h_J(x, y) = (P(x, y) - 1)n(x)n(y) \quad (2.18)$$

yields zero:

$$h_J(x, y)|F, Q\rangle = 0. \quad (2.19)$$

3 Ground state properties of the fermionic and bosonic $t - J$ model on a 2×2 plaquette

On a plaquette with 4 sites (2×2) the $t - J$ Hamiltonian can be diagonalized analytically [21]. It is interesting to compare the ground state energies $E^{(p)}(Q)$ and total spins $S^{(p)}(Q)$ in the sectors with Q spin-1/2 particles for the fermionic and bosonic version. It turns out

(1) they are the same in the sectors $Q = 0, 1, 3, 4$

$$E^{(p)}(0) = 0 \quad S^{(p)} = 0 \quad (3.1)$$

$$E_F^{(p)}(1) = -2t \quad (\alpha \geq 0) \quad S^{(p)} = \frac{1}{2} \quad (3.2)$$

$$E^{(p)}(3) = \begin{cases} -2t & (F) \quad S_F^{(p)} = \frac{3}{2} \\ -t\left(\alpha + \sqrt{\frac{\alpha^2}{4} + 3}\right) & (B) \quad S^{(p)} = \frac{1}{2} \\ -t\left(\frac{3\alpha}{2} + 1\right) & (C) \quad S^{(p)} = \frac{1}{2} \end{cases} \quad (3.3)$$

$$E^{(p)}(4) = -3t\alpha \quad (\alpha \geq 0) \quad S^{(p)} = 0 \quad (3.4)$$

where the α intervals (F), (B), (C) are defined by

$$(F) : 0 \leq \alpha \leq \alpha_F \left(\frac{3}{4}\right) = 0.262$$

$$(B) : \alpha_F \left(\frac{3}{4}\right) \leq \alpha \leq 2$$

$$(C) : \alpha \geq 2.$$

Ground state energies with index “F” correspond to ground states with maximal spin $S = Q/2$. In the sector with three particles ($Q = 3$) one finds the same energy

in the fermionic and bosonic version of the $t - J$ model for all $\alpha \geq 0$.

(2) they are different in the sector $Q = 2$

a) In the fermionic version,

$$\begin{aligned} E^{(p)}(2) &= -\frac{t}{2} \left(\alpha + \sqrt{\alpha^2 + 32} \right) \\ \alpha > 0, \quad S^{(p)}(2) &= 0 \end{aligned} \quad (3.5)$$

the ground state is nondegenerate and has total spin 0.

b) In the bosonic version:

$$\begin{aligned} E_F^{(p)}(2) &= -2t\sqrt{2} \\ 0 \leq \alpha \leq \alpha_F\left(\frac{1}{2}\right) &= \sqrt{2}, \quad S_F^{(p)}(2) = 1 \end{aligned} \quad (3.6)$$

$$\begin{aligned} E^{(p)}(2) &= -\frac{t}{2} \left(\alpha + \sqrt{\alpha^2 + 16} \right) \\ \alpha > \alpha_F\left(\frac{1}{2}\right), \quad S^{(p)}(2) &= 0 \end{aligned} \quad (3.7)$$

we find first a ferromagnetic ground state with maximal spin 1 and for larger α values degenerate spin 0 ground states.

Note that the ground state energy of the fermionic model (3.5) is below that of the bosonic model (3.6), (3.7) for $\alpha > 0$. The two energies coincide at $\alpha = 0$.

It is remarkable to note, that some of these similarities can be found on all square lattices $L \times L$, $L = 2, \sqrt{10}, 4$. Most important is the equality

$$\begin{aligned} E_f(Q = N - 1, \alpha, N) &= E_b(Q = N - 1, \alpha, N) \\ N &= L^2, \quad \alpha > 0 \end{aligned} \quad (3.8)$$

in the 1 hole sector for all $\alpha > 0$. The α dependence is shown in Figure 2.

In the sector with 2 particles ($Q = 2$) at $\alpha = 0$ the ground state energies are the same, but the ground state spin is always $S_f(Q) = 0$ in the fermionic, but $S_b(Q) = 1$ in the bosonic version, as already stated in (1.5).

4 The phase diagram of the bosonic $t - J$ model in two dimensions

In this paper we are mainly concerned with the question, whether the ground state with Q (spin-1/2) particles has maximal total spin $S = Q/2$. The answer to this question depends on the strength α of the spin exchange part (2.1) which prefers (for $\alpha > 0$) antiferromagnetic ordering with total spin $S = 0$.

Owing to the property (2.17) the expectation value of the spin exchange part

$$\varepsilon_J(\rho, \alpha) = \frac{1}{N} \langle E(Q, \alpha) | H_J | E(Q, \alpha) \rangle \quad (4.1)$$

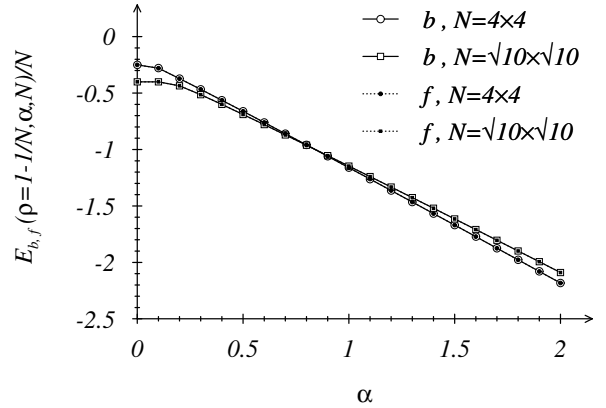


Fig. 2. α -dependence of ground state energies in the 1 hole sector of a $L \times L$ lattice for $\alpha > 0$. Bosonic and fermionic $t - J$ ground state energies for $L = \sqrt{10}, 4$ coincide.

can be considered as an order parameter; it vanishes if the ground state $|E(Q, \alpha)\rangle$ in the sector with Q particles has maximal spin $S = Q/2$, but is nonzero in all other cases.

We will demonstrate for the bosonic version of the $t - J$ model on $L \times L$ clusters that the ferromagnetic regime:

$$\varepsilon_J(\rho, \alpha) = 0 \quad \text{for } 0 \leq \alpha \leq \alpha_F(\rho = Q/N) \quad (4.2)$$

is bounded by a curve $\alpha = \alpha_F(\rho = Q/N)$ depending on the hole density $\delta = 1 - \rho$.

In our numerical calculations of the ground states $|E(Q, \alpha)\rangle$, $Q = N - 1, N - 2, \dots$ in the bosonic $t - J$ model with periodic boundary conditions, we only found ferromagnetic states $|F\rangle = |E_F(Q)\rangle$ for α small enough. Figure 3 shows the order parameter $\varepsilon_J(\rho, \alpha)$ on a 4×4 cluster in the charge sectors $Q = 3, 4, \dots, 15$.

The phase boundary $\alpha_F(\rho = Q/16)$ can be read off from those α -values where $\varepsilon_J(\rho, \alpha)$ changes first from 0 to a nonvanishing value. Obviously, $\alpha_F(\rho)$ is monotonically decreasing with ρ and vanishes for $\rho = 1$. Its ρ -dependence is shown in Figure 4.

For increasing values of Q , we observe more and more jumps in the order parameter $\varepsilon_J(\rho, \alpha)$. E.g. for $Q = 8$ we find three of these jumps. Each of them signals the transition to a new ground state with increasing values of α . The ground states differ in their total spin S which is denoted by the integer (Q even) and halfinteger (Q odd) numbers at the branches between two jumps. E.g. for $Q = 8$:

$$\begin{aligned} S = 4 & \quad \text{for } 0.0 \leq \alpha \leq 1.1 = \alpha_F(1/2) & |E_F\rangle \\ S = 2 & \quad \text{for } 1.2 \leq \alpha \leq 1.6 & |E_B\rangle \\ S = 1 & \quad \text{for } 1.7 \leq \alpha < 2.4 = \alpha_0(1/2) & |E_C\rangle \\ S = 0 & \quad \text{for } \alpha \geq 2.4 & |E_D\rangle. \end{aligned}$$

So far we were not able to localize the spin couplings for the ground state with total spin $S = 3$, which we expect to emerge in the interval $1.1 < \alpha < 1.2$. For increasing values of $\rho = Q/N$, $\alpha_F(\rho)$ approaches zero and it is more and more difficult to localize the α values for those

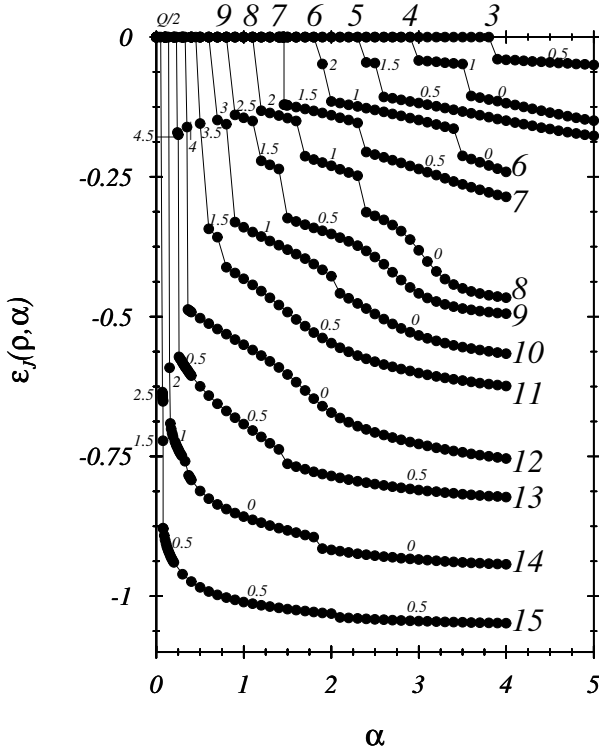


Fig. 3. Order parameter $\varepsilon_J(\rho, \alpha)$ for a $N = 4 \times 4$ cluster with charge sectors $Q = 3, \dots, 15$. The order parameter for each Q is turning away from zero at $\alpha = \alpha_F(\rho = Q/N)$. Small numbers denote the total spin of the bosonic $t - J$ ground states.

ground states with higher spin $S = Q/2 - 1, Q/2 - 2, \dots$. We performed higher resolved searches with step width $\Delta\alpha = 0.001$ and found e.g. in the $Q = 15$ sector the ground state with total spin $S = 1.5$ at $\alpha = 0.077$. In contrast, a similar search for the $S = 2.5$ ground state in the $Q = 7$ sector failed.

As an alternative we also tried to localize those α values:

$$\alpha_0(\rho), \quad \rho = \frac{Q}{N}, Q \text{ even}, S = 0$$

$$\alpha_{1/2}(\rho), \quad \rho = \frac{Q}{N}, Q \text{ odd}, S = \frac{1}{2}$$

where the total spin S of the ground state is first minimal. The curves $\alpha_0(\rho)$, $\alpha_{1/2}(\rho)$, and $\alpha_F(\rho)$ are shown in Figure 4.

All curves are monotonically decreasing with ρ . We have looked for the finite-size dependence of the curve $\alpha_F(\rho)$ by comparing results on $L \times L$ lattices for

$$L = \sqrt{10}, 4, \sqrt{18}, \sqrt{20}, \sqrt{26}.$$

On the larger systems with $L > 4$ we assumed that the ground state spin $S_b = Q/2$ is maximal and the curve $\alpha_F(\rho)$ was extracted from those α -values, where the spin is lowered first by one or two units. As can be seen from Figure 5, the finite-size effects can be rather well

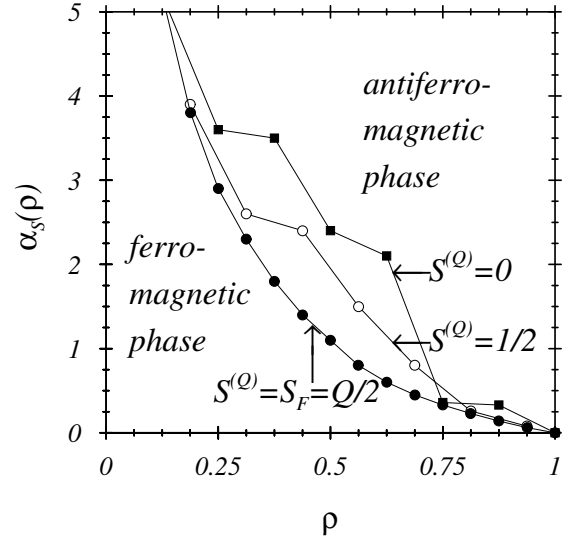


Fig. 4. ρ -dependence of the lines $\alpha_0(\rho)$, $\alpha_{1/2}(\rho)$, $\alpha_F(\rho)$ for a $N = 4 \times 4$ cluster.

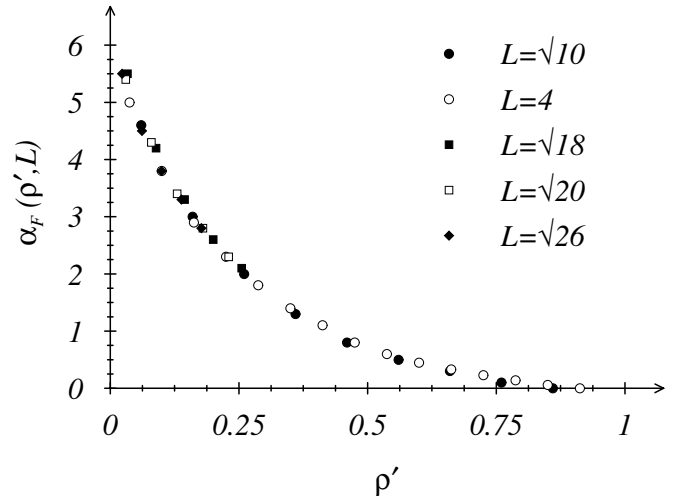


Fig. 5. Finite-size effects of $\alpha_F(\rho)$ by comparison of results on $L \times L$ lattices for $L = \sqrt{10}, 4, \sqrt{18}, \sqrt{20}, \sqrt{26}$. The finite-size effects has been accounted for by $\rho' = \rho - \frac{1.4}{N}$.

accounted for by an effective charge density

$$\rho' = \rho - \frac{c}{N}, \quad c = 1.4$$

such that $\alpha_F(\rho')$ scales with the system size $N = L \times L$.

A similar finite-size analysis for the curves $\alpha_0(\rho)$ and $\alpha_{1/2}(\rho)$, which mark the boundaries of the antiferromagnetic phase with minimal ground state spin demands simulations on much larger systems. They might be accessible with Monte Carlo techniques if it is possible to determine the ground state spin with high accuracy. The question of physical interest is, whether the transition region between the ferro- and antiferromagnetic phase in Figure 4 shrinks to zero in the thermodynamical limit.

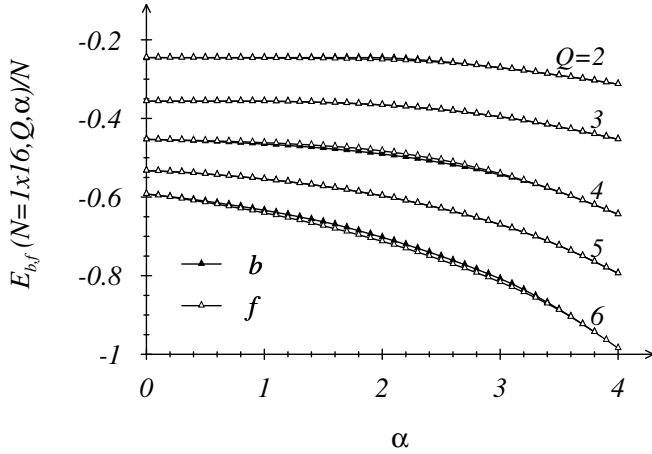


Fig. 6. Comparison of the bosonic and fermionic ground state energies per site ($E_{b,f}(N, Q, \alpha)/N$) for a one dimensional periodic chain $N = 16$ ($D = 1$) for charges $Q = 2, 3, 4, 5, 6$.

5 Comparison of the bosonic and fermionic $t-J$ model in one dimension

In one dimension the difference between the bosonic and fermionic version of the $t-J$ model can be absorbed in the boundary conditions. It turns out that the ground state energies with periodic boundary conditions on a $N = 16$ site system coincide for all α if Q is odd

$$E_b(Q, S_i(Q), \alpha, N) = E_f(Q, S_i(Q), \alpha, N). \quad (5.1)$$

For Q even one observes slight differences for $\alpha \simeq 2$, as can be seen from Figure 6.

At $\alpha = 0$ the equality (5.1) strictly holds for all Q -values and all the ground state energies satisfy the hole-particle symmetry (1.4).

Concerning the total spin of the ground state $S_i(Q)$ $i = b, f$ at $\alpha = 0$ we observe

- a maximal value $S_b(Q) = Q/2$ for all Q
- a maximal value $S_f(Q) = Q/2$ only for odd Q .

For Q even the ground states appear to be degenerate with different values for $S_f(Q)$ of the total spin. This leads to the “erratic” behaviour observed already in the two dimensional system $D = 2$.

Differences between the bosonic $t-J$ model for $D = 1$ and $D = 2$ become apparent in the variation of the order parameter (4.1) and the total spin $S_b(Q)$ with α . They are shown in Figure 7 for $D = 1$ and in Figure 3 for $D = 2$.

The main difference between the two cases $D = 2$ and $D = 1$ is the following:

The ferromagnetic domain $0 \leq \alpha \leq \alpha_F(\rho)$ – where the total spin is maximal – shrinks to zero in the 1D case. In particular for $Q > 3$, the boundary $\alpha_F(\rho)$ turns out to be below 0.1 already on finite chains.

6 Discussion and perspectives

In this paper we have demonstrated that the emergence of ferromagnetism in a hard-core model crucially depends on

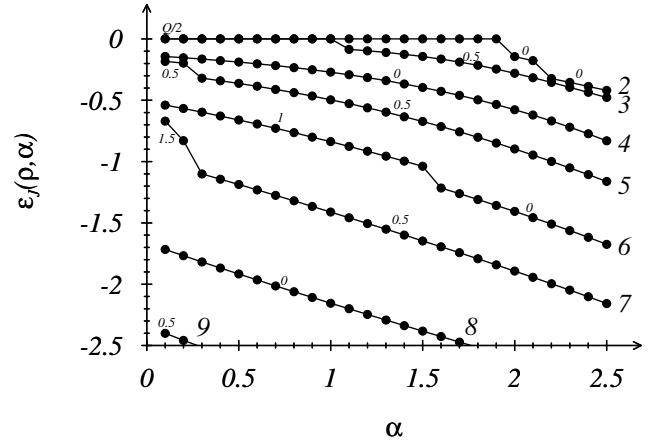


Fig. 7. Order parameter $\varepsilon_J(\rho, \alpha)$ for a $N = 1 \times 16$ cluster with charge sectors $Q = 3, \dots, 15$. Small numbers denote the total spin of the bosonic $t-J$ ground states.

the statistics of the constituent particles, i.e. whether they are fermions or bosons. We studied the $t-J$ model in the (traditional) fermionic (f) and in a bosonic (b) version and compared ground state energies $E_i(Q, S_i(Q), \alpha)$ and total spins $S_i(Q, \alpha)$, $i = f, b$, in both models. In particular, we found for $\alpha = 0$ that – in contrast to hard-core fermions – hard-core bosons (with spin 1/2) have a ferromagnetic ground state with maximal spin $S_b(Q) = Q/2$ in all charge sectors Q .

This means that the corresponding Hamiltonian H_t (2.10) leads intrinsically to a ferromagnetic interaction between spin-up and spin-down particles, as we have demonstrated first on finite square lattices $L \times L$, $L = 2, \sqrt{10}, 4$ numerically. A general proof is given in Appendix A.

The ground state energies for hard-core bosons and fermions behave differently on finite lattices. They show hole-particle symmetry (2.11) in the bosonic version, whereas this symmetry is not present in the fermionic version. We do not know whether this breaking is a finite-size effect and whether the ground state energies (Fig. 1) converge to each other in the thermodynamical limit. If we switch on the spin exchange coupling α [(2.3), (2.18)], the bosonic $t-J$ model in two dimensions ($D = 2$) develops a ferromagnetic regime $0 \leq \alpha \leq \alpha_F(\rho)$, where $\alpha_F(\rho)$ is monotonically decreasing with ρ . For $D = 1$, $\alpha_F(\rho)$ shrinks to zero.

Ferromagnetism is observed in the fermionic and bosonic $t-J$ model in the sectors $Q = 1$ (1 particle) and $Q = N - 1$ (1 hole). Here, the two $t-J$ models are unitary equivalent, which implies that the sign problem for the electrons can be absorbed in a unitary transformation [23]. In other words “electrons behave as bosons”. This behaviour can be found as well for hard-core electrons in 1D with periodic boundary conditions and Q odd.

On one hand all ground state energies $E_f(Q, \alpha = 0)$ $Q = 1, \dots, N - 1$ show the hole-particle symmetry (1.4) and are strictly identical with the corresponding quantities $E_b(Q, \alpha = 0)$ in the bosonic version. The ground state spins $S_f(Q)$ are maximal for Q odd, but this is not the

case for Q even, where the ground state appears to be degenerate with different total spins.

Let us finally comment on possible more realistic hard-core boson models for ferromagnetism. On one hand these bosons have to carry spin, on the other hand the spin statistics theorem demands integer spin for bosons. It is quite straightforward to extend the hopping Hamiltonian (2.10) to spin 1 bosons. In this case, the number of particles at site x :

$$n(x) = n_1(x) + n_{-1}(x) + n_0(x) = 0, 1 \quad (6.1)$$

has to be identified with the sum of spin 1, $-1, 0$ particles. Such a Hamiltonian is again invariant under particle-hole transformations. It defines a hard-core boson model with 4 degrees of freedom at each site: 3 for the spin-1 particles, 1 for the holes.

Appendix A: The ground state of the hopping Hamiltonian (2.10)

We want to present first a general proof that the ferromagnetic state with all spins up ($N_+ = Q$) is indeed a ground state of H_t (2.10) in the sector with Q particles. Here, the permutation operator $P(x, y)$ for a spin (+) and a hole can be represented:

$$P(x, y) = \frac{1}{2} \left(1 + \boldsymbol{\tau}(x)\boldsymbol{\tau}(y) \right) \quad (A.1)$$

in terms of Pauli matrices at neighbouring sites x and y which act on particle χ_+ and hole χ_h states as

$$\begin{aligned} \tau_3(x)\chi_j(x) &= (2n_+(x) - 1)\chi_j(x), \quad j = +, h \\ \chi_+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (A.2)$$

It is easy to verify that (2.10) in the sector $N_+ = Q$ can be mapped on the XX spin model in two dimensions:

$$H_t = -\frac{1}{2} \sum_{\langle x, y \rangle} \left(\tau_1(x)\tau_1(y) + \tau_2(x)\tau_2(y) \right). \quad (A.3)$$

The Hamiltonian (A.3) is invariant under the inversion of all “ τ spins” (A.2)

$$UH_tU^+ = H_t \quad U = \prod_x \tau_1(x) \quad (A.4)$$

which explains the hole-particle symmetry (2.11).

Note, that all the matrix elements in H_t have a negative sign. Therefore we can apply the Perron-Frobenius theorem [24], which states that the ground state $\psi_F(S_z = Q/2)$ is nondegenerate and that in the defining basis all components are greater than zero.

Since H_t conserves the total spin (2.15), the application of lowering operators

$$S_- = \frac{1}{2} \sum_{\langle x, y \rangle} \left(\sigma_1(x) - i\sigma_2(y) \right) \quad (A.5)$$

onto the ground state $\psi_F(S_z = Q/2)$ with eigenvalue $E_F(Q)$ yields eigenstates of H_t :

$$\begin{aligned} (S_-)^n \psi_F(S_z = Q/2) &= \langle \psi(Q/2) | S_+^n S_-^n | \psi(Q/2) \rangle^{-1/2} \\ &\quad \times \psi_F(S_z = Q/2 - n) \end{aligned} \quad (A.6)$$

$$H_t \psi_F(S_z = Q/2 - n) = E_F(Q) \psi_F(S_z = Q/2 - n) \quad (A.7)$$

in the sector with $S_z = Q/2 - n$ with the same eigenvalue $E_F(Q)$. Indeed these states are again ground states in the sector $S_z = Q/2 - n$ for the following two reasons:

- i) The state $\psi_F(S_z = Q/2 - n)$ has again only positive components in the defining basis in the sector $S_z = Q/2 - n$.
- ii) The matrix elements of H_t in the sector $S_z = Q/2 - n$ are all negative, such that the Perron-Frobenius theorem can be applied again: The ground state is nondegenerate and has only positive components. Therefore $\psi_F(S_z = Q/2 - n)$ must be the ground state for $S_z = Q/2 - n$.

Finally, we want to stress that all the states $\psi_F(S_z = Q/2 - n)$ are symmetric under all permutations:

$$\begin{aligned} (P(x, y) - 1) \left(n_+(x)n_-(y) + n_-(x)n_+(y) \right) \\ \times \psi_F(S_z = Q/2 - n) = 0 \end{aligned} \quad (A.8)$$

for the quantum numbers of the particles at sites x and y , which guarantees that

$$\begin{aligned} S^2 \psi_F(S_z = Q/2 - n) &= S(S+1) \psi_F(S_z = Q/2 - n) \\ S &= Q/2. \end{aligned} \quad (A.9)$$

Note added in proof

After this paper was accepted for publication, we learned that our bosonic results – at zero spin exchange coupling ($J = 0$) – are just a specific case of a more general theorem [25] of E. Eisenberg and E.H. Lieb, proving the existence of a fully polarized state among the ground states in the absence of explicit spin-dependent forces.

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